

QUOTIENT AND BLOW-UP OF AUTOMORPHISMS OF GRAPHS OF GROUPS

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ABSTRACT. In this paper we study the quotient and “blow-up” of graph-of-groups \mathcal{G} and of their automorphisms $H : \mathcal{G} \rightarrow \mathcal{G}$. We show that the existence of such a “blow-up” of $\overline{H} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ relative to a given family of “local” graph-of-groups isomorphisms $H_i : \mathcal{G}_i \rightarrow \mathcal{G}_i$ depends crucially on the H_i -conjugacy class of the correction term $\delta(\overline{E}_i)$ for any edge \overline{E}_i of $\overline{\mathcal{G}}$, where H -conjugacy is a new but natural concept introduced here. As an application we obtain a criterion as to whether a partial Dehn twist can be blown up relative to local Dehn twists to give an actual Dehn twist.

1. INTRODUCTION

Given a graph-of-groups \mathcal{G} based on a finite connected graph Γ , for any connected subgraph $\Gamma_0 \subset \Gamma$, we denote by \mathcal{G}_0 the restriction of \mathcal{G} to Γ_0 . There is a natural way to define a quotient graph of groups $\overline{\mathcal{G}} = \mathcal{G}/\mathcal{G}_0$ which is obtained by “contracting” \mathcal{G}_0 into a single vertex V_0 with vertex group $G_{V_0} \cong \pi_1(\mathcal{G}_0)$, thus giving rise to a canonical isomorphism $\Theta : \pi_1(\overline{\mathcal{G}}) \rightarrow \pi_1(\mathcal{G})$. By construction the quotient graph $\overline{\mathcal{G}}$ is *compatible* with the local graph-of-groups \mathcal{G}_0 , in the sense that for any edge E of $\overline{\mathcal{G}}$ with terminal vertex V_0 the canonical image of the edge group G_E in the vertex group $G_{V_0} \cong \pi_1(\mathcal{G}_0)$ is (up to conjugation) contained in one of the vertex groups of \mathcal{G}_0 .

This quotient concept extends naturally to an isomorphism H of \mathcal{G} which acts as identity on the underlying graph Γ . We can hence construct a quotient graph-of-groups isomorphism $\overline{H} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ which induces on the fundamental group an outer automorphism $\widehat{\overline{H}}$ conjugate to \widehat{H} via Θ , as shown in the diagram below.

$$\begin{array}{ccc} \pi_1(\mathcal{G}) & \xrightarrow{\widehat{H}} & \pi_1(\mathcal{G}) \\ \Theta \downarrow & & \downarrow \Theta \\ \pi_1(\overline{\mathcal{G}}) & \xrightarrow{\widehat{\overline{H}}} & \pi_1(\overline{\mathcal{G}}) \end{array}$$

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The restriction H_0 of H to \mathcal{G}_0 is called the *local graph-of-groups isomorphism* at V_0 . Of course, both the quotient graph-of-groups $\overline{\mathcal{G}}$ and the quotient isomorphism \overline{H} are also well-defined modulo more than one pairwise disjoint connected sub-graph-of-groups \mathcal{G}_i of \mathcal{G} .

In this paper we study the converse of this procedure, which is called the “blow-up” of a graph-of-groups isomorphism. We prove:

Theorem 1.1. *Let $\overline{H} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ be a graph-of-groups isomorphism which acts as identity on the graph $\overline{\Gamma}$ underlying $\overline{\mathcal{G}}$. Assume that for any vertex V_i of $\overline{\Gamma}$ the group isomorphism \overline{H}_{V_i} is induced by a compatible local graph-of-groups isomorphism $H_i : \mathcal{G}_i \rightarrow \mathcal{G}_i$. Then one can blow up $(\overline{H}, \overline{\mathcal{G}})$ via the family of isomorphisms H_i if and only if for any edge E of $\overline{\mathcal{G}}$ the correction term $\delta(E)$ is G_E -compatible and locally zero.*

Recall that for a graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$, for any edge e of \mathcal{G} with terminal vertex v , the correction term $\delta(e) \in G_v$ serves to make the edge and vertex isomorphisms H_e and H_v commute with the injective edge homomorphism $f_e : G_e \rightarrow G_v$. (See section 2 for details.)

Such a correction term $\delta(e)$ being locally zero plays a crucial role in the procedure of blowing up. In the special case where the local graph-of-groups isomorphism $H_v : \mathcal{G}_v \rightarrow \mathcal{G}_v$ is trivial, $\delta(e)$ is *locally zero* if it is conjugate to an element that has \mathcal{G}_v -length equal to zero. In the general case, conjugation in $\pi_1(\mathcal{G})$ (or rather in its ambient path group) needs to be replaced by H_v -conjugation, which we introduce and study in section 4.

In the last section of this paper we will apply Theorem 1.1 to the case of Dehn twist automorphisms of a free group F_n . Classically, a *Dehn twist* on a graph of groups $D = (\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ is defined by a family of *twistors* $(z_e)_{e \in E(\mathcal{G})}$, where each z_e is in the center of the edge group G_e . It turns out (see Proposition 3.8) that for free groups an alternative definition of a Dehn twist is given by graph-of-groups isomorphisms $H : \mathcal{G} \rightarrow \mathcal{G}$ where all edge groups of \mathcal{G} are trivial and H acts as identity on the underlying graph and on every vertex group of \mathcal{G} .

Inspired by this alternative definition, we define in section 7 a *partial Dehn twist* $D : \mathcal{G} \rightarrow \mathcal{G}$ with $\pi_1(\mathcal{G}) \cong F_n$, relative to some family of vertices V_1, \dots, V_m of the underlying graph $\Gamma(\mathcal{G})$, which differs from the above notion in that on these “exceptional vertices” V_i the *local* automorphism induced by D may be non-trivial.

Of particular interest is the case where these non-trivial local automorphisms are all Dehn twists themselves. This occurs naturally if one quotients a given Dehn twists modulo a family of pairwise disjoint subgraphs. The converse direction, however, is far less obvious, and the desired blow-up Dehn twist doesn’t always exist. We prove here:

Corollary 1.2. *A partial Dehn twist $\overline{D} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$, relative to a family of local Dehn twists D_{V_i} associated to the vertices V_i of \mathcal{G} , can be blown up to give an actual Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$, if every edge E_i of $\overline{\mathcal{G}}$ has correction term $\delta(E_i)$*

which is locally zero (i.e. with respect to the Dehn twist $D_{V_i} : \mathcal{G}_{V_i} \rightarrow \mathcal{G}_{V_i}$ given at the endpoint V_i of E_i).

It turns out that, when all the local Dehn twists are *efficient*, the condition that all $\delta(e)$'s being locally zero is not just sufficient but also necessary for the conclusion of Corollary 1.2. This will be used in [13] as a crucial ingredient of the proof that every linearly growing outer automorphism of a finitely generated free group F_n is (up to taking powers) a Dehn twist automorphism. It is also a vital ingredient in our algorithmic approach to the question which automorphisms of F_n are induced by surface homeomorphisms.

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2. BASICS OF GRAPHS OF GROUPS AND THEIR ISOMORPHISMS

In this section, we recall some basic knowledge about graph-of-groups as well as their isomorphisms. Most of our notations are taken from [1]; we refer the readers to [3], [7] and [8] for more detailed informations and discussions.

2.1. Basic Conventions.

Unless otherwise stated, a *graph* refers to a finite, non-empty, connected graph in the sense of Serre (cf. [3]).

We recall the notations here. For a graph Γ , we denote by $V(\Gamma)$, $E(\Gamma)$ its *vertex set* and *edge set* respectively. For an edge $e \in E(\Gamma)$, we denote \bar{e} its *inverse*, $\tau(e)$ its *terminal vertex* and $\tau(\bar{e})$ its *initial vertex*.

Notice in particular that our graph Γ is non-oriented. An *orientation* of Γ refers to a subset $E^+(\Gamma) \subset E(\Gamma)$ such that $E^+(\Gamma) \cup \bar{E}^+(\Gamma) = E(\Gamma)$ and $E^+(\Gamma) \cap \bar{E}^+(\Gamma) = \emptyset$, where $\bar{E}^+(\Gamma) = \{\bar{e} \mid e \in E^+(\Gamma)\}$.

For an arbitrary group G , we denote by $ad_x : G \rightarrow G$ the inner automorphism defined by element $x \in G$, namely $ad_x(g) = xgx^{-1}$ for all $g \in G$.

2.2. Graphs of Groups.

Definition 2.1. A *graph-of-groups* \mathcal{G} is defined by

$$\mathcal{G} = (\Gamma, (G_v)_{v \in V(\Gamma)}, (G_e)_{e \in E(\Gamma)}, (f_e)_{e \in E(\Gamma)})$$

where:

- (1) Γ is a graph, called the *underlying graph*;
- (2) each G_v is a group, called the *vertex group* of v ;
- (3) each G_e is a group, called the *edge group* of e , and we require $G_e = G_{\bar{e}}$ for every $e \in E(\Gamma)$;
- (4) for each $e \in E(\Gamma)$, the map $f_e : G_e \rightarrow G_{\tau(e)}$ is an injective *edge homomorphism*.

For a graph-of-groups \mathcal{G} , we usually denote by $\Gamma(\mathcal{G})$ the graph underlying it. The vertex set of $\Gamma(\mathcal{G})$ is denoted by $V(\mathcal{G})$ while the edge set is denoted by $E(\mathcal{G})$.

Definition 2.2. For a graph-of-groups \mathcal{G} , its *word group* $W(\mathcal{G})$ is the free product of vertex groups and the free group generated by *stable letters* $(t_e)_{e \in E(\Gamma)}$, i.e. $W(\mathcal{G}) = *(G_v)_{v \in V(\Gamma)} * F(\{t_e; e \in E(\Gamma)\})$.

The *path group* (sometimes also called *Bass group*) of \mathcal{G} is defined by $\Pi(\mathcal{G}) = W(\mathcal{G})/R$, where R is the normal subgroup determined by the following relations:

- ◊ $t_e = t_{\bar{e}}^{-1}$, for every $e \in E(\Gamma)$;
- ◊ $f_{\bar{e}}(g) = t_e f_e(g) t_{\bar{e}}^{-1}$, for every $e \in E(\Gamma)$ and every $g \in G_e$.

Remark 2.3. A word $w \in W(\mathcal{G})$ can always be written in the form $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$ ($q \geq 0$), where each $t_i \in F(\{t_e; e \in E(\Gamma)\})$ stands for the stable letter of the edge e_i and each $r_i \in *(G_v)_{v \in V(\Gamma)}$.

The sequence (t_1, t_2, \dots, t_q) is called the *path type* of w , the number q is called the *path length* of w . In this case, we say that $e_1 e_2 \dots e_q$ is the path underlying w . Two path types (t_1, t_2, \dots, t_q) and $(t'_1, t'_2, \dots, t'_s)$ are said to be same if and only if $q = s$ and $t_i = t'_i$ for each $1 \leq i \leq q$.

Definition 2.4. Let $w \in W(\mathcal{G})$ be a word of the form $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$. The word w is said to be *connected* if $r_0 \in G_{\tau(\bar{e}_1)}$, $r_q \in G_{\tau(e_q)}$, and $\tau(e_i) = \tau(\bar{e}_{i+1})$, $r_i \in G_{\tau(e_i)}$, for $i = 1, 2, \dots, q-1$.

Moreover, if w is connected and $\tau(e_q) = \tau(\bar{e}_1)$, we say that w is a *closed connected word issued at the vertex* $\tau(e_q)$.

Definition 2.5. Let $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q \in W(\mathcal{G})$, w is said to be *reduced* if it satisfies:

- ◊ if $q = 0$, then $w = r_0$ isn't equal to the unit element;
- ◊ if $q > 0$, then whenever $t_i = t_{i+1}^{-1}$ for some $1 \leq i \leq q-1$ we have $r_i \notin f_{e_i}(G_{e_i})$.

Moreover the word w is said to be *cyclically reduced* if it is reduced and if $q > 0$ and $t_1 = t_q^{-1}$, then $r_q r_0 \notin f_{e_q}(G_{e_q})$.

We recall the following facts.

Proposition 2.6. For any graph-of-groups \mathcal{G} , the following holds:

- (1) Every non-trivial element of $\Pi(\mathcal{G})$ can be represented as a reduced word.
- (2) Every reduced word is a non-trivial element in $\Pi(\mathcal{G})$.
- (3) If $w_1, w_2 \in W(\mathcal{G})$ are two reduced words representing the same element in $\Pi(\mathcal{G})$, then w_1 and w_2 are of the same path type. In particular, w_2 is connected if and only if w_1 is connected.

In fact, suppose $w_1 = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$ and $w_2 = r'_0 t'_1 r'_1 \dots r'_{q-1} t'_q r'_q$, then there exist elements $h_i \in G_{e_i}$ ($i = 1, 2, \dots, q$) such that:

$r'_0 = r_0 f_{\bar{e}_1}(h_1)$; $r'_i = f_{e_i}(h_i) r_i f_{\bar{e}_{i+1}}(h_{i+1}^{-1})$ for ($i = 1, 2, \dots, q-1$); and $r'_q = f_{e_q}(h_q) r_q$.

Definition 2.7. For any $v_0 \in V(\Gamma)$ the *fundamental group based at* v_0 , denoted by $\pi_1(\mathcal{G}, v_0)$, consists of the elements in $\Pi(\mathcal{G})$ that are closed connected words issued at v_0 .

For a vertex $w_0 \in V(\Gamma)$ different from v_0 , we have $\pi_1(\mathcal{G}, v_0) \cong \pi_1(\mathcal{G}, w_0)$. In fact, let $W \in \Pi(\mathcal{G})$ be a connected word with underlying path from v_0 to w_0 . The restriction of $ad_W : \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ to $\pi_1(\mathcal{G}, w_0)$ induces an isomorphism from $\pi_1(\mathcal{G}, w_0)$ to $\pi_1(\mathcal{G}, v_0)$. Sometimes we write $\pi_1(\mathcal{G})$ when the choice of basepoint doesn't make a difference.

2.3. Graph-of-Groups Isomorphism.

Definition 2.8. Let $\mathcal{G}_1, \mathcal{G}_2$ be two graphs of groups. Denote $\Gamma_1 = \Gamma(\mathcal{G}_1)$ and $\Gamma_2 = \Gamma(\mathcal{G}_2)$. An isomorphism $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a tuple of the form

$$H = (H_\Gamma, (H_v)_{v \in V(\Gamma_1)}, (H_e)_{e \in E(\Gamma_1)}, (\delta(e))_{e \in E(\Gamma_1)})$$

where

- (1) $H_\Gamma : \Gamma_1 \rightarrow \Gamma_2$ is a graph isomorphism;
- (2) $H_v : G_v \rightarrow G_{H_\Gamma(v)}$ is a group isomorphism, for any $v \in V(\Gamma_1)$;
- (3) $H_e = H_{\bar{e}} : G_e \rightarrow G_{H_\Gamma(e)}$ is a group isomorphism, for any $e \in E(\Gamma_1)$;
- (4) for every $e \in E(\Gamma_1)$, the *correction term* $\delta(e) \in G_{\tau(H_\Gamma(e))}$ is an element such that

$$H_{\tau(e)} f_e = ad_{\delta(e)} f_{H_\Gamma(e)} H_e.$$

Remark 2.9. A graph-of-groups isomorphism $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces an isomorphism $H_* : \Pi(\mathcal{G}_1) \rightarrow \Pi(\mathcal{G}_2)$ defined on the generators by:

$$\begin{aligned} H_*(g) &= H_v(g), \text{ for } g \in G_v, v \in V(\Gamma_1); \\ H_*(t_e) &= \delta(\bar{e}) t_{H_\Gamma(e)} \delta(e)^{-1}, \text{ for } e \in E(\Gamma_1). \end{aligned}$$

It's easy to verify by computation that H_* preserves the relations $t_e t_{\bar{e}} = 1$ for any $e \in E(\mathcal{G})$ and $f_{\bar{e}}(g) = t_e f_e(g) t_e^{-1}$, for any $e \in E(\mathcal{G})$ and $g \in G_e$.

Furthermore, the restriction of H_* to $\pi_1(\mathcal{G}_1, v)$, where $v \in V(\Gamma_1)$, is also an isomorphism, denoted by $H_{*v} : \pi_1(\mathcal{G}_1, v) \rightarrow \pi_1(\mathcal{G}_2, H_\Gamma(v))$.

As in [1], we define the *outer isomorphism* induced by a group isomorphism $f : G_1 \rightarrow G_2$ as the equivalence class

$$\hat{f} = \{ad_g f : G_1 \rightarrow G_2 \mid g \in G_2\}.$$

Hence H_{*v} induces an outer isomorphism $\hat{H}_{*v} : \pi_1(\mathcal{G}_1, v) \rightarrow \pi_1(\mathcal{G}_2, H_\Gamma(v))$.

Observe that when choosing a different vertex v_1 as basepoint, we may choose a word $W \in \Pi(\mathcal{G}_1)$ with underlying path from v_1 to v to obtain the following commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathcal{G}_1, v) & \xrightarrow{H_{*v}} & \pi_1(\mathcal{G}_2, H_\Gamma(v)) \\ ad_W \downarrow & & ad_{H_*(W)} \downarrow \\ \pi_1(\mathcal{G}_1, v_1) & \xrightarrow{H_{*v_1}} & \pi_1(\mathcal{G}_2, H_\Gamma(v_1)) \end{array}$$

By Lemma 2.2 and Lemma 3.10 in [1], \hat{H}_{*v} determines an outer isomorphism $\hat{H}_{*v_1} : \pi_1(\mathcal{G}_1, v_1) \rightarrow \pi_1(\mathcal{G}_2, H_\Gamma(v_1))$.

In this sense, we observe that $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces an outer isomorphism $\hat{H} : \pi_1(\mathcal{G}_1) \rightarrow \pi_1(\mathcal{G}_2)$ which doesn't depend on the choice of basepoint.

Remark 2.10 (Composition, Inverse). For two graph-of-groups isomorphisms $H' : \mathcal{G}_1 \rightarrow \mathcal{G}_2$, $H'' : \mathcal{G}_2 \rightarrow \mathcal{G}_3$, the *composition* of H' and H'' is an isomorphism $H''H' = H : \mathcal{G}_1 \rightarrow \mathcal{G}_3$ given (for any $v \in V(\Gamma_1)$, $e \in E(\Gamma_1)$) precisely by: $H_\Gamma = H''_\Gamma H'_\Gamma$; $H_v = H''_{H'_\Gamma(v)} H'_v$; $H_e = H''_{H'_\Gamma(e)} H'_e$; $\delta(e) = H''_{\tau(H'_\Gamma(e))}(\delta'(e))\delta''(H'_\Gamma(e))$. Moreover, H satisfies $H_* = H''_* H'_*$ and $\widehat{H} = \widehat{H}'' \widehat{H}'$.

For any graph-of-groups isomorphism $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ the *inverse* isomorphism is $H^{-1} : \mathcal{G}_2 \rightarrow \mathcal{G}_1$, which satisfies $H_*^{-1} = (H^{-1})_*$ and $\widehat{H}^{-1} = \widehat{H}^{-1}$, is defined (for all $v \in V(\Gamma_2)$, $e \in E(\Gamma_2)$) by: $(H^{-1})_\Gamma = (H_\Gamma)^{-1}$; $(H^{-1})_v = (H_{H_\Gamma^{-1}(v)})^{-1}$; $(H^{-1})_e = (H_{H_\Gamma^{-1}(e)})^{-1}$; $\delta^{-1}(e) = H_{H_\Gamma^{-1}(\tau(e))}^{-1}(\delta(H_\Gamma^{-1}(e))^{-1})$.

2.4. A Natural Equivalence Between Graphs Of Groups.

Suppose $\mathcal{G}, \mathcal{G}'$ are two graphs-of-groups, \mathcal{G}' equals to \mathcal{G} everywhere except that for one edge $e_0 \in E(\mathcal{G})$, $f'_{e_0} = ad_{g_0^{-1}} \circ f_{e_0}$, where g_0 is an element in $G_{\tau(e_0)}$. Then there is a natural isomorphism between \mathcal{G} and \mathcal{G}' .

More concretely, define $H_0 : \mathcal{G} \rightarrow \mathcal{G}'$ by the rules:

- (1) $H_{0,\Gamma} = id_{\Gamma(\mathcal{G})}$;
- (2) $H_{0,v} = id_{G_v}$, for all $v \in V(\mathcal{G})$; $H_{0,e} = id_{G_e}$ for all $e \in E(\mathcal{G})$;
- (3) $\delta_0(e_0) = g_0$ and $\delta_0(e) = id$ when $e \neq e_0$.

Then it's easy to verify that H_0 is a well defined graph-of-groups isomorphism since the additional compatibility requirement holds automatically for all edges $e_0 \neq e \in \Gamma_0$, and for e_0 we have

$$H_{0,\tau(e_0)} \circ f_{e_0} = f_{e_0} = ad_{g_0} \circ f'_{e_0} = ad_{\delta_0(e_0)} \circ f'_{e_0} = ad_{\delta_0(e_0)} \circ f'_{e_0} \circ H_{0,e_0}.$$

The above isomorphism gives rise to a natural notion of “equivalent” graphs-of-groups, where the equivalence relation is generated by isomorphisms of the above type H_0 as elementary equivalence. This notion of “equivalent” graph-of-groups, although not really established in the literature, is natural, in that it preserves (up to canonical isomorphisms) the fundamental group. It also shows up in the prime feature of graph-of-groups, meaning Bass-Serre theory: Given a group G that acts on a (simplicial) tree T , for the associated graph-of-groups decomposition $G \cong \pi_1(\mathcal{G}_T)$ the “quotient” graph-of-groups \mathcal{G}_T of T modulo G is only well defined up to precisely this equivalence relation.

Lemma 2.11. *Let $H' : \mathcal{G}' \rightarrow \mathcal{G}'$ be equal to H everywhere except that $\delta'(e_0) = H_{\tau(e_0)}(w_{e_0})^{-1}\delta(e_0)w_{e_0}$, where e_0 is an edge in E and $w_{e_0} \in G_{\tau(e_0)}$ for all $e_0 \in E$.*

Then H' is a well defined graph-of-groups automorphism which is conjugate to H via H_0 .

Proof. To verify H' is a well defined graph-of-groups isomorphism, it's sufficient to verify that $H_{\tau(e_0)} \circ f'_{e_0} = ad_{\delta'(e_0)} \circ f'_{e_0} \circ H_{e_0}$.

For every $g \in G_{e_0}$,

$$\begin{aligned}
 ad_{\delta'(e_0)} \circ f'_{e_0} \circ H_{e_0}(g) &= H_{\tau(e_0)}(w_{e_0})^{-1} \delta(e_0) w_{e_0} f'_{e_0}(H_{e_0}(g)) w_{e_0}^{-1} \delta(e_0)^{-1} H_{\tau(e_0)}(w_{e_0}) \\
 &= H_{\tau(e_0)}(w_{e_0})^{-1} \delta(e_0) f_{e_0}(H_{e_0}(g)) \delta(e_0)^{-1} H_{\tau(e_0)}(w_{e_0}) \\
 &= H_{\tau(e_0)}(w_{e_0})^{-1} H_{\tau(e_0)}(f_{e_0}(g)) H_{\tau(e_0)}(w_{e_0}) \\
 &= H_{\tau(e_0)}(w_{e_0}^{-1} f_{e_0}(g) w_{e_0}) \\
 &= H_{\tau(e_0)} \circ f'_{e_0}(g)
 \end{aligned}$$

Moreover, we have the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{G} & \xrightarrow{H} & \mathcal{G} \\
 H_0 \downarrow & & \downarrow H_0 \\
 \mathcal{G}' & \xrightarrow{H'} & \mathcal{G}'
 \end{array}$$

The only non-trivial part is to verify that the following equation holds for all edges:

$$H_{0, H_{\Gamma_0}(\tau(e))}(\delta(e)) \delta_0(H_{\Gamma}(e)) = H'_{H_{\Gamma_0}(\tau(e))}(\delta_0(e)) \delta'(H_{\Gamma_0}(e))$$

When $e \neq e_0$, this equation is equivalent to $\delta(e) = \delta'(e)$ which holds automatically by definition.

When $e = e_0$,

$$\begin{aligned}
 Left &= \delta(e_0) \delta_0(e_0) = \delta(e_0) w_{e_0} \\
 Right &= H'_{\tau(e_0)}(\delta_0(e_0)) \delta'(e_0) = H'_{\tau(e_0)}(w_{e_0}) \delta'(e_0) = H_{\tau(e_0)}(w_{e_0}) \delta'(e_0) = \delta(e_0) w_{e_0}
 \end{aligned}$$

Hence we have the equation holds for all edges and that $H' = H_0 \circ H \circ H_0^{-1}$, which implies $\widehat{H}' = \widehat{H}_0 \circ \widehat{H} \circ \widehat{H}_0^{-1}$. \square

3. DEHN TWISTS

3.1. Classical Dehn Twist. We first recall the classical definition of a Dehn twist.

Definition 3.1 (Dehn Twist). An automorphism D of a graph-of-groups \mathcal{G} is called a *Dehn twist* if it satisfies:

- (1) $D_{\Gamma} = id_{\Gamma}$;
- (2) $D_v = id_{G_v}$, for all $v \in V(\Gamma)$;
- (3) $D_e = id_{G_e}$, for all $e \in E(\Gamma)$;
- (4) for each G_e , there is an element $\gamma_e \in Z(G_e)$ such that $\delta(e) = f_e(\gamma_e)$, where $Z(G_e)$ denotes the center of G_e .

We denote a Dehn twist defined as above by $D = D(\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$

Remark 3.2 (Twistor). Given a Dehn twist $D = D(\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$, we define the *twistor* of an edge $e \in E(\Gamma)$ by setting $z_e = \gamma_{\bar{e}} \gamma_e^{-1}$. Then for any edge e we have $z_e \in Z(G_e)$ and $z_{\bar{e}} = \gamma_e \gamma_{\bar{e}}^{-1} = z_e^{-1}$.

Remark 3.3. The induced automorphism $D_* : \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ is defined on generators as follows:

$$\begin{aligned} D_*(g) &= g, \text{ for } g \in G_v, v \in V(\Gamma); \\ D_*(t_e) &= t_e f_e(z_e), \text{ for every } e \in E(\Gamma). \end{aligned}$$

In particular, the induced automorphism on the fundamental group, $D_{*v} : \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}, v)$ where $v \in V(\Gamma)$, is called a *Dehn Twist automorphism*.

Definition 3.4. In general, a group automorphism $\varphi : G \rightarrow G$ is said to be a *Dehn twist automorphism* if it is represented by a graph-of-groups Dehn twist. In other words, there exists a graph-of-groups \mathcal{G} , a Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$, and an isomorphism $\theta : G \rightarrow \pi_1(\mathcal{G}, v)$ such that $\varphi = \theta^{-1} \circ D_{*v} \circ \theta$.

In this case the induced outer automorphism $\hat{\varphi} : G \rightarrow G$ is called a *Dehn twist outer automorphism*.

Remark 3.5. In general, for a Dehn twist automorphism $\varphi : G \rightarrow G$, its Dehn twist representative is not unique.

In particular, if G is a free group, the graph-of-groups Dehn twist which represents φ is only unique up to isomorphism after making it efficient (see subsection 3.3). For a Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$ which represents φ , if θ_1, θ_2 are two isomorphisms such that $\varphi = \theta_i^{-1} \circ D_{*v} \circ \theta_i$ ($i = 1, 2$), then $\theta_1 \circ \theta_2^{-1}, \theta_2 \circ \theta_1^{-1} \in C(D_{*v})$, where $C(D_{*v})$ denotes the centralizer of D_{*v} .

Remark 3.6 (Dehn Twist Homeomorphisms on Surfaces). Recall that on an oriented surface S (possibly with finitely many boundary components), a (left) *Dehn twist* about a *simple closed curve* α refers to a homeomorphism $T_\alpha : S \rightarrow S$ such that

- ◇ on a regular neighborhood N_α of α , for any $x \in N_\alpha$

$$T_\alpha(x) = \varphi \circ T \circ \varphi^{-1}(x)$$

where $T : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$ is the (left) *Twist map* which is defined by $T(\theta, t) = (\theta + 2\pi t, t)$, $\varphi : S^1 \times [0, 1] \rightarrow N_\alpha$ is an orientation preserving homeomorphism.

- ◇ T_α leaves the rest of S invariant.

A *multiple Dehn twist homeomorphism* h on S is obtained by performing such Dehn twist surgery n_i -times ($1 \leq i \leq k$) about a family of non-intersecting simple closed curves $\mathcal{C} = \{c_1, c_2, \dots, c_k\}$. Then the induced automorphism $h_* : \pi_1(S) \rightarrow \pi_1(S)$ is a *Dehn twist automorphism* (see Definition 3.4).

Furthermore, as explained in [13], if \mathcal{C} consists of essential simple closed curves which are pairwise non-homotopic, non-intersecting (note that these two conditions can always be achieved without changing the given homeomorphism h), then such homeomorphism h can be represented naturally, as constructed, by an efficient Dehn twist.

Proposition 3.7 (Proposition 5.4 [1]). *Suppose \mathcal{G} is a graph-of-groups which satisfies that for every edge e there is an element $r_e \in G_{\tau(e)}$ with*

$$f_e(G_e) \cap r_e f_e(G_e) r_e^{-1} = \{1\}.$$

Then two Dehn twists $D = (\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$, $D' = (\mathcal{G}, (\gamma'_e)_{e \in E(\mathcal{G})})$ determine the same outer automorphism of $\pi_1(\mathcal{G})$ if and only if $z_e = z'_e$ for all $e \in E(\Gamma)$.

This proposition shows that in many situations a Dehn twist on a given graph-of-groups is uniquely determined by its twistors. Thus sometimes we may define a Dehn twist by its twistors $(z_e)_{e \in E(\Gamma)}$ (for each $e \in E(\Gamma)$, $z_e \in Z(G_e)$ and $z_{\bar{e}} = z_e^{-1}$). In this case, we may conversely define:

$$\gamma_e = \begin{cases} z_e^{-1}, & e \in E^+(\Gamma) \\ 1, & e \in E^-(\Gamma). \end{cases}$$

3.2. General Dehn Twist.

3.2.1. The principal goal of this subsection is to establish the following proposition:

Proposition 3.8. *Let \mathcal{G} be a graph-of-groups such that the edge groups G_e are trivial for all $e \in E(\mathcal{G})$. Let $H : \mathcal{G} \rightarrow \mathcal{G}$ be an automorphism such that*

- ◊ H_Γ acts on $\Gamma(\mathcal{G})$ as identity;
- ◊ $H_v : G_v \rightarrow G_v$ is identity, for all $v \in V(\mathcal{G})$.

Then H induces a Dehn twist automorphism on the fundamental group of \mathcal{G} .

We will first show that the notion of a Dehn twist can be understood in a slightly more general context, as explained now. The proof of Proposition 3.8 will be explained in Remark 3.11 as a corollary of the following proposition.

Definition 3.9. Let \mathcal{G} be a graph-of-groups. An automorphism $D : \mathcal{G} \rightarrow \mathcal{G}$ is called a *general Dehn twist* if

- ◊ $D_\Gamma = id_\Gamma$;
- ◊ $D_v = id_{G_v}$, for all vertices;
- ◊ $D_e = id_{G_e}$, for all edges;
- ◊ $\delta(e) \in C(f_e(G_e))$, where $C(f_e(G_e))$ denotes the centralizer of $f_e(G_e)$ in $G_{\tau(e)}$, for all $e \in E(\Gamma)$.

Proposition 3.10. *The definition of classical Dehn twist and general Dehn twist are “equivalent” in the following sense:*

- (1) *Every classical Dehn twist is a general Dehn twist.*
- (2) *To every general Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$ we can canonically associate a classical Dehn twist $D' : \mathcal{G}' \rightarrow \mathcal{G}'$ and an isomorphism $\theta_v : \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}', v')$ (for any vertex v for \mathcal{G} and a corresponding vertex v' for \mathcal{G}') such that $D'_{*v'} \circ \theta_v = \theta_v \circ D_{*v}$.*

Proof. From a classical Dehn twist to a general Dehn twist is obvious by definition.

For the converse direction, we first consider a special case: Let $D : \mathcal{G} \rightarrow \mathcal{G}$ be a general Dehn twist with all correction terms defined in the classical case, except for a single edge e_0 : we suppose $\delta(e_0) \in C(f_{e_0}(G_{e_0})) \subset G_{\tau(e_0)}$ which is not contained in $Z(f_{e_0}(G_{e_0}))$. Then we define a classical Dehn twist $D' : \mathcal{G}' \rightarrow \mathcal{G}'$ obtained as follows:

- ◇ The graph-of-groups \mathcal{G}' is obtained from \mathcal{G} by subdividing the edge e_0 into e'_0 and e''_0 , with $\iota(e'_0) = \iota(e_0)$, $\tau(e'_0) = v_0 = \iota(e''_0)$ and $\tau(e''_0) = \tau(e_0)$, by setting $G_{e'_0} = G_{e_0}$, $G_{e''_0} = G_{v_0} = \langle f_{e_0}(G_{e_0}), \delta(e_0) \rangle$, and by defining the edge homomorphisms by $f_{e'_0} = f_{e_0}$, $f_{\bar{e}'_0} = f_{\bar{e}_0}$; $f_{e''_0} = id$, $f_{\bar{e}''_0} = id$.
- ◇ The Dehn twist D' is defined by setting $D'_{e'_0} = id_{G_{e'_0}}$, $D'_{e''_0} = id_{G_{e''_0}}$, $D'_{v_0} = id_{G_{v_0}}$, and by choosing $\delta'(\bar{e}'_0) = \delta(\bar{e}_0)$, $\delta'(e'_0) = id$, $\delta'(\bar{e}''_0) = id$, $\delta'(e''_0) = \delta(e_0)$. The data for D' equal those of D everywhere else.

From $\delta(e_0) \in C(f_{e_0}(G_{e_0}))$ we derive directly $\delta(e_0) \in Z(G_{e''_0})$. Thus D' is a classical Dehn twist as given in Definition 3.4.

Consider the homomorphism $\theta : \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G}')$ defined on generators via $t_{e_0} \mapsto t_{e'_0} t_{e''_0}$ and $g \mapsto g$ otherwise. It is easy to see that θ is injective.

Then D' is “equal” to D in the following sense: for every vertex $v \neq v_0$ from \mathcal{G}' there is a corresponding vertex for \mathcal{G} which we also call v . The restriction of θ to $\pi_1(\mathcal{G}, v)$ defines a map $\theta_v : \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}', v)$ which is easily seen to be an isomorphism.

Furthermore, θ gives rise to the following diagram:

$$\begin{array}{ccc} \pi_1(\mathcal{G}, v) & \xrightarrow{D_{*v}} & \pi_1(\mathcal{G}, v) \\ \theta_v \downarrow & & \downarrow \theta_v \\ \pi_1(\mathcal{G}', v) & \xrightarrow{D'_{*v}} & \pi_1(\mathcal{G}', v) \end{array}$$

This diagram is easily seen to be commutative; the only non-trivial argument is given by:

$$\begin{aligned} \theta_v \circ D_{*v}(t_{e_0}) &= \theta_v(\delta(\bar{e}_0) t_{e_0} \delta(e_0)^{-1}) = \delta(\bar{e}_0) t_{e'_0} t_{e''_0} \delta(e_0)^{-1} \\ &= D'_{*v}(t_{e'_0}) D'_{*v}(t_{e''_0}) = D'_{*v}(t_{e'_0} t_{e''_0}) \\ &= D'_{*v} \circ \theta_v(t_{e_0}). \end{aligned}$$

In the general case, where D is a general Dehn twist which may have more than one correction term defined in the “non-classical way”, we may apply the above treated special case repeatedly to each of the “non-classical” correction terms, to eventually obtain a classical Dehn twist. \square

Remark 3.11. Proposition 3.8 can thus be deduced as a corollary of Proposition 3.10 given the fact that when the edge group G_e is trivial the centralizer of its image in $G_{\tau(e)}$ is the entire group, i.e. $Z(f_e(G)_e) = G_{\tau(e)}$.

3.3. Efficient Dehn Twist.

Let $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ be a Dehn twist. Two edges e_1 and e_2 with common terminal vertex v are called

- ◊ *positively bonded*, if there exist $n_1, n_2 \geq 1$ such that $f_{e_1}(z_{e_1}^{n_1})$ and $f_{e_2}(z_{e_2}^{n_2})$ are conjugate in G_v .
- ◊ *negative bonded*, if there exist $n_1 \geq 1, n_2 \leq 1$ such that $f_{e_1}(z_{e_1}^{n_1})$ and $f_{e_2}(z_{e_2}^{n_2})$ are conjugate in G_v .

For the rest of this subsection, we always assume for a graph-of-groups \mathcal{G} its fundamental group $\pi_1(\mathcal{G})$ is free and of finite rank $n \geq 2$.

Definition 3.12 (Efficient Dehn twist [1]). A Dehn twist $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ is said to be *efficient* if :

The graph-of-groups \mathcal{G} satisfies

- (1) \mathcal{G} is *minimal*: if $v = \tau(e)$ is a valence-one vertex, then the edge homomorphism $f_e : G_e \rightarrow G_v$ is not surjective.
- (2) There is no *invisible vertex*: there is no valence-two vertex $v = \tau(e_1) = \tau(e_2)$ ($e_1 \neq e_2$) such that both edge maps $f_{e_i} : G_{e_i} \rightarrow G_v$ ($i = 1, 2$) are surjective.
- (3) No *proper power*: if $r^p \in f_e(G_e)$ ($p \neq 0$) then $r \in f_e(G_e)$, for all $e \in E(\Gamma)$.

And together with the collection of twistors $(z_e)_{e \in E(\Gamma)}$, it also satisfies:

- (4) No *unused edge*: for every $e \in E(\Gamma)$, the twistor $z_e \neq 1$ (or equivalently $\gamma_e \neq \gamma_{\bar{e}}$).
- (5) If $v = \tau(e_1) = \tau(e_2)$, then e_1 and e_2 are not positively bonded.

It's demonstrated in [1] that every Dehn twist automorphism can be transformed algorithmically into an efficient Dehn twist, which we restate in Proposition 3.13.

Proposition 3.13. *For every Dehn twist $D = D(\mathcal{G}, (z_e)_{e \in E(\Gamma)})$, there exists an efficient Dehn twist $D' = D(\mathcal{G}', (z_e)_{e \in E(\Gamma')})$ and an isomorphism between fundamental groups $\rho : \pi_1(\mathcal{G}, w) \rightarrow \pi_1(\mathcal{G}', w')$, where $w \in V(\Gamma)$, $w' \in V(\Gamma')$ are properly choosen vertices, such that $\widehat{D'}\widehat{\rho} = \widehat{\rho}\widehat{D}$.*

Proof. Follows directly from Propostion 8.4 and Theorem 8.5 in [1]. □

4. H-CONJUGATION

Definition 4.1. Let H be an isomorphism of a graph-of-groups \mathcal{G} . Let w_1, w_2 be words in the Bass group $\Pi(\mathcal{G})$. The word w_1 is said to be *H-conjugate* to w_2 if there exists $w \in \Pi(\mathcal{G})$ such that $w_1 = ww_2H_*(w)^{-1}$.

Lemma 4.2. *H-conjugation is a well defined equivalence relation on $\Pi(\mathcal{G})$.*

Proof. Reflexivity and symmetricity are obvious.

Transitivity is clear because if $w_1 = ww_2H_*(w)^{-1}$ and $w_2 = w'w_3H_*(w')^{-1}$, then $w_1 = ww'w_3H_*(ww')^{-1}$. □

Denote by $[w]_H$ the set which consists of all elements in $\Pi(\mathcal{G})$ that are H -conjugate to w . We call $[w]_H$ the H -conjugacy class of w .

Recall that the *path length* of a word $w \in \Pi(\mathcal{G})$ equals to the number of edges the path underlying w crosses. We denote the path length of w by $\|w\|_{\mathcal{G}}$.

Definition 4.3. A reduced word $w \in \Pi(\mathcal{G})$ is said to be H -minimal if, for every $w_0 \in \Pi(\mathcal{G})$, $\|w_0 w H_*(w_0)^{-1}\|_{\mathcal{G}} \geq \|w\|_{\mathcal{G}}$.

Since $\|w\|_{\mathcal{G}}$ is a natural number, one has:

Lemma 4.4. For every reduced word $w \in \Pi(\mathcal{G})$ there exists $w_0 \in \Pi(\mathcal{G})$ such that $w_0 w H_*(w_0)^{-1}$ is H -minimal.

Definition 4.5. Let $w \in \Pi(\mathcal{G})$ be a reduced word in the form of $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$, w is said to be H -reduced if its cannot be shortened by the elementary operation $w \mapsto w_1 = (r_0 t_1)^{-1} w H_*(r_0 t_1)$, i.e. $\|w\|_{\mathcal{G}} = \|w_1\|_{\mathcal{G}}$.

It can be shown that a reduced word $w \in \Pi(\mathcal{G})$ is H -minimal if and only if it is H -reduced.

Proposition 4.6. Let $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q \in \Pi(\mathcal{G})$, then (at least) one of the following is H -reduced:

$$\begin{aligned} w_1 &= (r_0 t_1)^{-1} w H_*(r_0 t_1); \\ w_2 &= (r_1 t_2)^{-1} w_1 H_*(r_1 t_2); \\ &\dots \\ w_i &= (r_{i-1} t_i)^{-1} w_{i-1} H_*(r_{i-1} t_i); \\ &\dots \\ w_q &= (r_{q-1} t_q)^{-1} w_{q-1} H_*(r_{q-1} t_q). \end{aligned}$$

Proof. For two reduced words $x, y \in \Pi(\mathcal{G})$, we always have

$$|\|x\|_{\mathcal{G}} - \|y\|_{\mathcal{G}}| \leq \|xy\|_{\mathcal{G}} \leq \|x\|_{\mathcal{G}} + \|y\|_{\mathcal{G}}.$$

Let $w_0 \in \Pi(\mathcal{G})$ be a reduced word. Observe that, if w_0 doesn't end with $t_1^{-1}x$, where $x \in *(G_v)_{v \in V(\mathcal{G})}$, then $\|w_0 w\|_{\mathcal{G}} = \|w_0\|_{\mathcal{G}} + \|w\|_{\mathcal{G}}$.

Moreover since $\|w_0\|_{\mathcal{G}} = \|H_*(w_0^{-1})\|_{\mathcal{G}}$, we have

$$\begin{aligned} \|w_0 w H_*(w_0^{-1})\|_{\mathcal{G}} &\geq \|w_0 w\|_{\mathcal{G}} - \|H_*(w_0^{-1})\|_{\mathcal{G}} \\ &= \|w_0\|_{\mathcal{G}} + \|w\|_{\mathcal{G}} - \|H_*(w_0^{-1})\|_{\mathcal{G}} = \|w\|_{\mathcal{G}}. \end{aligned}$$

Similarly for shortening w backwards.

We always have $\|w_i\|_{\mathcal{G}} = \|w_{i-1}\|_{\mathcal{G}}$ or $\|w_i\|_{\mathcal{G}} = \|w_{i-1}\|_{\mathcal{G}} - 2$, for $1 < i \leq q$. When $\|w_i\|_{\mathcal{G}} = \|w_{i-1}\|_{\mathcal{G}}$, we have w_{i-1}, w_i are already H -reduced. \square

Remark 4.7. If $w' = r_0 t_1 r_1 \dots r_{k-1} t_k r_k \in [w]_H$ is H -reduced, then $w'_1 = (r_0 t_1)^{-1} w' H_*(r_0 t_1), \dots, w'_i = (r_{i-1} t_i)^{-1} w'_{i-1} H_*(r_{i-1} t_i), \dots, w'_k = (r_{k-1} t_k)^{-1} w'_{k-1} H_*(r_{k-1} t_k)$ are also H -reduced.

Moreover, since H_* preserves the path lengths of reduced words, we also have $H_*(w') = w'^{-1}w'H_*(w')$, $H_*^{-1}(w') = H_*^{-1}(w')w'w'^{-1}$ are H -reduced.

Hence $\bigcup_{i=-\infty}^{+\infty} \{H_*^i(w'), H_*^i(w'_1), \dots, H_*^i(w'_k)\}$ covers all path types of H -reduced words in $[w]_H$.

Remark 4.8. From Lemma 4.4 we obtain directly:

(a) For every element $w \in \Pi(\mathcal{G})$, there exists a well defined H -length:

$$\|w\|_{\mathcal{G},H} = \min\{\|gwH_*(g)^{-1}\|_{\mathcal{G}} \mid g \in \Pi(\mathcal{G})\}.$$

(b) $\|w\|_{\mathcal{G},H} = \|w\|_{\mathcal{G}}$ if and only if w is H -reduced.

Definition 4.9. A reduced word $w \in \Pi(\mathcal{G})$ is called H -zero if and only if its H -length equals to zero, i.e. $\|w\|_{\mathcal{G},H} = 0$.

Lemma 4.10. Let $H_i : \mathcal{G}_i \rightarrow \mathcal{G}_i$ ($i = 1, 2$) be two graph-of-groups isomorphisms, $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a graph-of-groups isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{H_1} & \mathcal{G}_1 \\ H \downarrow & & \downarrow H \\ \mathcal{G}_2 & \xrightarrow{H_2} & \mathcal{G}_2 \end{array}$$

For any $w_0 \in \Pi(\mathcal{G}_1)$, $H_*(w_0) \in \Pi(\mathcal{G}_2)$ is H_2 -zero if and only if w_0 is H_1 -zero.

Proof. From one side, the word w_0 is H_1 -zero implies there exists $w_1 \in \Pi(\mathcal{G}_1)$ such that $\|w_1w_0H_{1*}(w_1^{-1})\|_{\mathcal{G}_1,H_1} = 0$, i.e. $w_1w_0H_{1*}(w_1^{-1}) \in G_{v_1}$, for some $v_1 \in V(\mathcal{G}_1)$.

Hence $H_*(w_1w_0H_{1*}(w_1^{-1})) \in G_{H(v_1)}$, whence has path length zero.

Since $H_*(w_1), H_*(w_0), H_*(H_{1*}(w_1^{-1})) \in \Pi(\mathcal{G}_2)$, also we observed that

$$\begin{aligned} H_*(w_1w_0H_{1*}(w_1^{-1})) &= H_*(w_1)H_*(w_0)H_*(H_{1*}(w_1^{-1})) \\ &= H_*(w_1)H_*(w_0)H_{2*}(H_*(w_1^{-1})) \\ &= H_*(w_1)H_*(w_0)H_{2*}(H_*(w_1)^{-1}). \end{aligned}$$

Thus $H_*(w_0)$ is H_2 -zero.

Symmetrically, we also have w_0 is H_1 -zero if $H_*(w_0)$ is H_2 -zero. \square

5. QUOTIENT GRAPH-OF-GROUPS ISOMORPHISM

Let \mathcal{G} be a graph-of-groups and \mathcal{G}_0 be a sub-graph-of-groups of it. By sub-graph-of-groups we simply mean the restriction of \mathcal{G} to a connected subgraph¹. Denote $\Gamma = \Gamma(\mathcal{G})$ and $\Gamma_0 = \Gamma(\mathcal{G}_0)$, by definition Γ_0 is a connected subgraph of Γ .

¹ Be aware that this definition of sub-graph-of-groups is different from the one in [8].

We will denote $\overline{\mathcal{G}} = \mathcal{G}/\mathcal{G}_0$ the quotient graph-of-groups of \mathcal{G} by \mathcal{G}_0 , which we define now in detail:

The graph underlying $\overline{\mathcal{G}}$, denoted by $\overline{\Gamma} = \Gamma/\Gamma_0$ with $V(\overline{\Gamma}) = V(\Gamma) \setminus V(\Gamma_0) \cup \{V_0\}$ and $E(\overline{\Gamma}) = E(\Gamma) \setminus E(\Gamma_0)$, is obtained precisely by contracting Γ_0 into a vertex V_0 through the map:

$$\begin{aligned} q : \Gamma &\rightarrow \overline{\Gamma} \\ x &\mapsto V_0 \quad \text{if } x \in E(\Gamma_0) \text{ or } V(\Gamma_0) \\ x &\mapsto X \quad \text{otherwise} \end{aligned}$$

On $\Gamma \setminus \Gamma_0$, q maps x , where x is either an edge or a vertex, to its natural correspondence in $\overline{\Gamma}$. We denote it by the same but capitalized letter $X \in \overline{\Gamma}$, to avoid confusion.

We choose a vertex

$$(5.1) \quad P_0 \in V(\Gamma_0)$$

as base point and set $G_{V_0} = \pi_1(\mathcal{G}, P_0)$.

For every $v \in V(\Gamma)$ and $V = q(v) \neq V_0$, we set $G_V = G_v$. For every $e \in E(\Gamma)$ and $E = q(e) \in E(\overline{\Gamma})$, let $G_E = G_e$, and $f_E = f_e$ if $\tau(E) \neq V_0$. If $\tau(E) = V_0$, we choose a word

$$(5.2) \quad \gamma_e \in \Pi(\mathcal{G}_0) \subset \Pi(\mathcal{G})$$

from P_0 to $\tau(e) \in V(\Gamma_0)$ and define the edge homomorphism of E to be $f_E := ad_{\gamma_e} \circ f_e$.

Our formal construction of quotient graph-of-group automorphism depends on the choice of base point P_0 in Γ_0 and of “connected words” γ_e for all edges with $\tau(e) \in V(\Gamma_0)$, as in (5.2). However, we will show in Remark 5.3 that the induced outer automorphism is unique up to conjugacy.

Let $H : \mathcal{G} \rightarrow \mathcal{G}$ be a graph-of-groups isomorphism which acts as identity on the graph Γ . The restriction of H to \mathcal{G}_0 , denoted by $H_0 = H|_{\mathcal{G}_0} : \mathcal{G}_0 \rightarrow \mathcal{G}_0$, is called *local graph-of-groups isomorphism*.

Define $\overline{H} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ through setting:

- (1) $\overline{H}_{V_0} = H_{0,*P_0} : \pi_1(\mathcal{G}_0, P_0) \rightarrow \pi_1(\mathcal{G}_0, P_0)$;
- (2) $\delta(E) = H_*(\gamma_e)\delta(e)\gamma_e^{-1}$, for all E such that $\tau(E) = V_0$;
- (3) \overline{H} equals to H on the rest of $\overline{\mathcal{G}}$.

Lemma 5.1. \overline{H} is a well defined graph-of-groups isomorphism which induces an outer automorphism conjugate to \widehat{H} by the isomorphism $\theta : \pi_1(\mathcal{G}) \rightarrow \pi_1(\overline{\mathcal{G}})$.

Proof. To verify that \overline{H} is a well defined graph-of-groups isomorphism, it's sufficient to verify that $\overline{H}_{V_0} \circ f_E = ad_{\delta(E)} \circ f_E \circ \overline{H}_E$ for all $E = q(e)$ with $\tau(E) = V_0$. Denote $v = \tau(e) \in V(\Gamma_0)$.

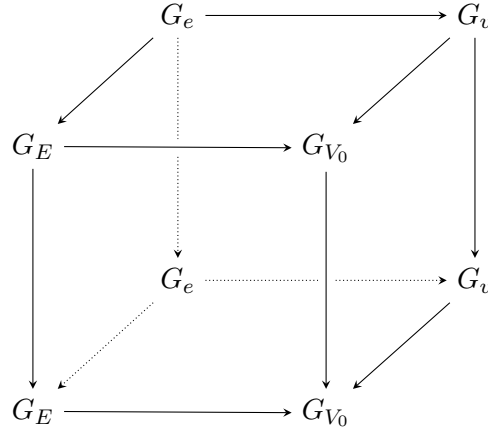
Observed first that for every $g \in G_v$, since H_0 induces an isomorphism on $\Pi(\mathcal{G}_0)$, we have

$$\overline{H}_{V_0}(\gamma_e g \gamma_e^{-1}) = H_{0*}(\gamma_e) H_v(g) H_{0*}(\gamma_e^{-1}) = H_*(\gamma_e) H_v(g) H_*(\gamma_e^{-1}).$$

For every $g \in G_E$,

$$\begin{aligned}
 ad_{\delta(E)} \circ f_E \circ \overline{H}_E(g) &= H_*(\gamma_e)\delta(e)\gamma_e^{-1}f_E(\overline{H}_E(g))\gamma_e\delta(e)^{-1}H_*(\gamma_e^{-1}) \\
 &= H_*(\gamma_e)\delta(e)f_e(H_e(g))\delta(e)^{-1}H_*(\gamma_e^{-1}) \\
 &= H_*(\gamma_e)H_v(f_e(g))H_*(\gamma_e^{-1}) \\
 &= \overline{H}_{V_0}(\gamma_e f_e(g)\gamma_e^{-1}) \\
 &= \overline{H}_{V_0} \circ f_E(g).
 \end{aligned}$$

Visually we may see that every face of the cube demonstrated below commutes (up to inner automorphisms).



Define $\theta : \pi_1(\overline{\mathcal{G}}, V_0) \rightarrow \pi_1(\mathcal{G}, P_0)$ a map which is defined via:

$$\begin{aligned}
 \theta : \pi_1(\overline{\mathcal{G}}, V_0) &\rightarrow \pi_1(\mathcal{G}, P_0) \\
 t_E &\mapsto t_e\gamma_e^{-1} \quad \text{if } \tau(E) = V_0, \tau(\overline{E}) \neq V_0 \\
 t_E &\mapsto \gamma_e t_e \gamma_e^{-1} \quad \text{if } \tau(E) = \tau(\overline{E}) = V_0 \\
 \theta &\text{ acts as identity elsewhere}
 \end{aligned}$$

Through the identification $G_{V_0} = \pi_1(\mathcal{G}, P_0)$ we see directly that θ define an isomorphism. In particular, since θ acts on vertex groups as identity, we also have, for $E \in E(\overline{\mathcal{G}})$ with $\tau(E) = V_0$ but $\tau(\overline{E}) \neq V_0$:

$$\begin{aligned}
 \theta \circ \overline{H}_*(t_E) &= \theta(\delta(\overline{E})t_E\delta(E)^{-1}) = \delta(\overline{E})t_e\gamma_e^{-1}\delta(E)^{-1} \\
 &= \delta(\overline{e})t_e\delta(e)^{-1}H_*(\gamma_e)^{-1} \\
 &= H_*(t_e\gamma_e^{-1}) \\
 &= H_* \circ \theta(t_E).
 \end{aligned}$$

Similarly for $E \in E(\overline{\mathcal{G}})$ with $\tau(E) = \tau(\overline{E}) = V_0$.

Therefore $\theta \circ \overline{H}_{*V_0} = H_{*P_0} \circ \theta$, and hence $\widehat{\overline{H}}$ and \widehat{H} are conjugate to each other. \square

Remark 5.2. Note in particular, for all E with $\tau(E) = V_0$, we have by definition that $\delta(E)$ is H_0^{-1} -conjugate to an element with H -length equal to zero, and hence is H_0^{-1} -zero.

Remark 5.3. Suppose that for some edge e with $\tau(e) \in V(\Gamma_0)$, one chooses a different γ'_e from P_0 to $\tau(e)$, hence f' is different from f_E by $ad_{\gamma'_e \gamma_e^{-1}}$. By Lemma 2.11, the new resulting graph-of-groups isomorphism $\overline{H}' : \overline{\mathcal{G}}' \rightarrow \overline{\mathcal{G}}'$ is conjugate to \overline{H} .

When choosing vertex Q_0 different from P_0 as basepoint, one would have the graph-of-groups isomorphism constructed accordingly conjugate to $H_{*Q_0} \in \widehat{H}$.

Therefore the outer automorphism induced by the quotient graph-of-groups isomorphism is unique up to conjugacy.

Remark 5.4. We may apply this quotient procedure above on several disjoint connected subgraph-of-groups of \mathcal{G} and obtain an analogue conclusion that the quotient graph-of-groups isomorphism \overline{H}' induces an outer automorphism conjugate to \widehat{H} .

6. BLOWING UP GRAPH-OF-GROUPS AUTOMORPHISM

For simplicity of the presentation, we only give the blow-up construction at a single vertex. However (for example through iterating this procedure), one can generalize the technique described in this section directly to a blow-up construction at several vertices simultaneously.

Let $\overline{H} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ be a graph-of-groups automorphism, and let V_0 be a vertex of the underlying graph $\overline{\Gamma}$. We need the following assumptions:

- (1) \overline{H} acts on the graph $\overline{\Gamma}$ as identity;
- (2) For the chosen vertex $V_0 \in V(\overline{\mathcal{G}})$, there exist a graph-of-groups automorphism $H_0 : \mathcal{G}_0 \rightarrow \mathcal{G}_0$ and group isomorphism $\theta_0 : G_{V_0} \rightarrow \pi_1(\mathcal{G}_0, P_0)$ such that $\theta_0 \circ H_{V_0} = H_{0,*P_0} \circ \theta_0$ for some $P_0 \in V(\mathcal{G}_0)$;
- (3) *compatibility requirement for graph-of-groups:*
For any edge E of $\Gamma(\overline{\mathcal{G}})$ with $\tau(E) = V_0$ there exist some $v(E) \in V(\mathcal{G}_0)$ and a word $\gamma_E \in \Pi(\mathcal{G}_0)$ from P_0 to $v(E)$ such that $\theta_0 \circ f_E(G_E) \subset \gamma_E G_{v(E)} \gamma_E^{-1}$;
- (4) *compatibility requirement for isomorphism:*
The word γ_E also satisfies that $H_{0*}(\gamma_E^{-1})\theta_0(\delta(E))\gamma_E \in G_{v(E)}$.

Remark 6.1. Note that,

- (1) the compatibility requirement for isomorphism implies that for all $\delta(E)$, where $\tau(E) = V_0$, $\theta_0(\delta(E))$'s are H_0^{-1} -zero. We then say that $\delta(E)$'s are *locally zero*.

Conversely, in order to derive the compatibility requirement for isomorphism for a correction term $\delta(E)$ which is locally zero one needs the additional hypothesis that $\delta(E)$ is G_E -compatible: by this

we means that there is a connected word $\gamma_E \in \Pi(\mathcal{G})$ which satisfies both condition (3) and (4) above.

- (2) for each E with $\tau(E) = V_0$, there may exist more then one pair of $(v(E), \gamma_E)$ such that the above conditions hold.

Then we may construct the *blowing up* of $(\overline{\mathcal{G}}, \overline{H})$ as the following:

- (1) $\Gamma(\mathcal{G})$ is obtained from $\Gamma(\mathcal{G})$ and $\Gamma(\mathcal{G}_0)$ by replacing every edge E_i with terminal vertex $\tau(E_i) = V_0$ by an edge e_i with same initial vertex as E_i and terminal vertex $v(E)$. The analogous replacement is done for \overline{E} . Let $G_e = G_E$ and define $f_e(g) = \gamma_E^{-1} \theta_0(f_E(g)) \gamma_E$, for every $g \in G_e$. All other information remains.
- (2) $H : \mathcal{G} \rightarrow \mathcal{G}$ equals to H_0, \overline{H} when restricted on $\mathcal{G}_0, \overline{\mathcal{G}} \setminus \{V_0\}$ respectively except that $\delta(e) = H_{0*}(\gamma_E^{-1} \theta_0(\delta(E))) \gamma_E$.

Proposition 6.2. *The automorphism $H : \mathcal{G} \rightarrow \mathcal{G}$, defined as above, is a well defined graph-of-groups isomorphism which induces outer automorphism \widehat{H} conjugate to $\widehat{\overline{H}}$.*

Proof. First of all, following the compatibility of graph-of-groups, for every edge $e \in E(\Gamma)$ we have $f_e(G_e) = \gamma_E^{-1} \theta_0(f_E(G_E)) \gamma_E \subset G_{v(E)}$. Hence \mathcal{G} is a well defined graph-of-groups.

Furthermore observe that, for $g \in G_e$

$$\begin{aligned} ad_{\delta(e)} \circ f_e \circ H_e(g) &= \delta(e) f_e(H_e(g)) \delta(e)^{-1} \\ &= H_{0*}(\gamma_E^{-1} \theta_0(\delta(E))) \gamma_E f_e(H_e(g)) \gamma_E^{-1} \theta_0(\delta(E)^{-1}) H_{0*}(\gamma_E) \\ &= H_{0*}(\gamma_E^{-1} \theta_0(\delta(E) f_E(H_e(g)) \delta(E)^{-1})) H_{0*}(\gamma_E) \\ &= H_{0*}(\gamma_E^{-1} \theta_0(H_{V_0}(f_E(g)))) H_{0*}(\gamma_E) \\ &= H_{0*}(\gamma_E^{-1} H_{V_0}(\theta_0(f_E(g)))) H_{0*}(\gamma_E) \\ &= H_{0*}(\gamma_E^{-1} \theta_0(f_E(g)) \gamma_E) = H_{0*}(f_e(g)) = H_{v_E} \circ f_e(g) \end{aligned}$$

Hence H is a well defined graph-of-groups isomorphism.

Notice that G_{V_0} is an invariant free factor of $\pi_1(\overline{\mathcal{G}}, V_0)$, therefore $\theta : \pi_1(\overline{\mathcal{G}}, V_0) \rightarrow \pi_1(\mathcal{G}, P_0)$ defined by

- ◊ $\theta|_{G_{V_0}} = \theta_0$;
- ◊ $t_E \mapsto t_e \gamma_E^{-1}$, when E is an edge with $\tau(E) = V_0$ but $\tau(\overline{E}) \neq V_0$;
- ◊ $t_E \mapsto \gamma_E^{-1} t_e \gamma_E^{-1}$, if E is an edge with $\tau(E) = \tau(\overline{E}) = V_0$;
- ◊ θ acts as identity elsewhere.

Then θ is an isomorphism such that $\theta \circ \widehat{\overline{H}} = \widehat{H} \circ \theta$. In particular, we have $\theta \circ \overline{H}_{*V_0} = H_{*P_0} \circ \theta$. \square

7. PARTIAL DEHN TWIST CASE

Let \mathcal{G} be a graph-of-groups such that for some vertex $v_0 \in V(\mathcal{G})$, all edges with $\tau(e) = v_0$ have trivial edge groups.

Let $H : \mathcal{G} \rightarrow \mathcal{G}$ be a graph-of-groups isomorphism such that

- ◇ $H_\Gamma : \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G})$ is identity;
- ◇ $H_e = id_{G_e}$, for all edges;
- ◇ $H_v = id_{G_v}$, for all vertices $v \neq v_0$, and H_{v_0} is group isomorphism;
- ◇ $\delta(e) \in C(f_e(G_e))$ for all edges.

Definition 7.1. The graph-of-groups automorphism defined as above is called a *partial Dehn twist* relative to the vertex v_0 .

Remark 7.2. The condition that all edges ending at v_0 have trivial edge groups guarantees that the additional compatibility requirements before Remark 6.1 hold automatically for an arbitrary group isomorphism H_{v_0} .

Corollary 7.3. Let $H : \mathcal{G} \rightarrow \mathcal{G}$ be a partial Dehn twist with respect to a vertex v_0 . Assume that the following conditions hold:

- ◇ the group automorphism H_{v_0} is a Dehn twist automorphism represented by a Dehn twist (D_0, \mathcal{G}_0) and an isomorphism $\theta_0 : G_{v_0} \rightarrow \pi_1(\mathcal{G}_0, v)$, where $v \in V(\mathcal{G}_0)$. (We call (D_0, \mathcal{G}_0) the local Dehn twist associated to v_0 .)
- ◇ moreover for all edges $e \in E(\mathcal{G})$ with $\tau(e) = v_0$, we have $\theta_0(\delta(e))$'s are D_0^{-1} -zero. (We then say that $\delta(e)$ is locally zero.)

Then we can blow up (H, \mathcal{G}) at v_0 via (D_0, \mathcal{G}_0) to obtain a Dehn twist.

Proof. Since all edges terminating at v_0 has trivial edge group, the compatibility requirement of graph-of-groups holds automatically. Since we also have $\delta(e)$ is D_0^{-1} -zero for all such edges, therefore the compatibility requirement of isomorphism also holds.

We can hence blowing up (H, \mathcal{G}) with (D_0, \mathcal{G}_0) at vertex v_0 following the instructions in the section of blowing up graph-of-groups. Then it's easy to verify by definition that the obtained graph-of-groups isomorphism is Dehn twist. \square

Let \mathcal{G} be a graph-of-groups such that there exists a subset $V_0 \subset V(\mathcal{G})$ which satisfies that for all $v \in V_0$, G_v is a free group of finite rank $k \geq 2$ and all edges with $\tau(e) = v$ have trivial edge groups.

Similarly to Definition 7.1, we define *partial Dehn twist relative to V_0* to be a graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$ such that

- ◇ $H_\Gamma : \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G})$ is identity;
- ◇ $H_e = id_{G_e}$, for all edges;
- ◇ $H_v = id_{G_v}$, for all vertices $v \notin V_0$, and H_v is group isomorphism for all $v \in V_0$;
- ◇ $\delta(e) \in C(f_e(G_e))$ for all edges.

In analogy to Corollary 7.3 we have the following corollary.

Corollary 7.4. A partial Dehn twist (D, \mathcal{G}) relative to a family of vertices V_0 of $\Gamma(\mathcal{G})$ can be blown up via a family of local Dehn twists for to each $v \in V_0$ to give an actual Dehn twist if the correction terms $\delta(e)$ of D are locally zero, for any edge e of $\Gamma(\mathcal{G})$ with terminal vertex contained in V_0 .

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